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Non-linear capillary instability of a liquid jet

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A third-order theory has been developed to study capillary instability of a liquid jet. The result shows that the asymmetrical development of an initially sinusoidal wave is a non-linear effect with generation of higher harmonics as well as feedback into the fundamental. The growth of the surface wave is found to depend explicitly on the dimensionless initial amplitude of the disturbance and the dimensionless wave-number k of the wave. For the same initial disturbance, the wave is found to have a maximum growth rate at $k = 0{\cdot}7$ in agreement with the linearized theory. For the same wave-number, the growth is proportional to the initial amplitude of the disturbance. The cut-off wave-number and the fundamental frequency (or growth rate for the unstable case) of the wave for a given k are found to be different from the linearized theory. Furthermore, at the cut-off wave-number, the present theory shows the disturbance experiences a growth which is proportional to t^2. The excellent agreement between Donnelly & Glaberson's experiment and Rayleigh's linearized theory is found to be due to their method of measurement.

1. Introduction

The capillary instability of a circular liquid jet was studied in the nineteenth century by Savart, Plateau and Rayleigh. Their experimental and analytical work are summarized by Rayleigh (1945). Neglecting the effect of surrounding air, Rayleigh showed that only axisymmetrical surface disturbances with wavelengths larger than π or $3{\cdot}14$ diameters of the jet would grow. The surface waves grow as $e^{\omega t}$, where

$$\omega^2 = \frac{T}{R^3\rho} kR \frac{I_1(kR)}{I_0(kR)} [1 - k^2R^2]. \tag{1.1}$$

In (1.1), T is the surface tension, ρ is the density of liquid, R is the undisturbed radius of the jet, k is the wave-number and I_0, I_1 are the modified Bessel functions. The dispersion curve of ω versus k shows a maximum growth rate at $kR = 0{\cdot}697$ and a cut-off wavelength at $kR = 1$.

Because of its technological importance, a great amount of work has been done on jet instability. The majority of the work is, however, concentrated on the study of jet break-up lengths and corresponding break-up time.

Although Rayleigh introduced the idea of using external disturbances to induce jet instability of different frequencies, this idea was not applied by others to study the dispersion curve until recently. Crane, Birch & McCormack (1964) studied jet instability by introducing mechanical vibration using a highly flexible electronically driven electrical vibrator. Their measured growth rate only agreed qualitatively with Rayleigh's result. They noted the non-sinusoidal behaviour of the surface deformation but no explanation was given for its occurrence. Donnelly & Glaberson (1966) made a careful study of the growth rate of a liquid jet by introducing sinusoidal disturbances of different wavelengths using a loudspeaker driven by an audio oscillator. By measuring the difference between the neck and the swell of the surface deformation as a function of time they were able to show that the growth rate ω was a constant and agreed well with the theory of Rayleigh. The agreement is good using the above method of measurement even when the surface wave is definitely non-sinusoidal and grows within one wavelength of the disengagement of the drop from the jet. Thus they concluded that no non-linear effect came into play in the break-up of the jet and that the non-sinusoidal surface deformation was due to the presence of higher harmonics in the vibrating system as was first suggested by Rayleigh.

The linearized analysis shows that the surface with radius r grows as

$$r = R + \eta_0 \cos kz\, e^{\omega t}, \tag{1.2}$$

where η_0 is the amplitude of the initial disturbance. Equation (1.2) holds to first order of η_0 because mass is conserved only to first order if R is the unperturbed radius. Thus, for finite surface waves with a disturbance of the above form to hold, R has to be a decreasing function of time. This means that even in the above form the neck and swell will grow at a different rate contrary to the linearized theory. On the other hand, in an experimental and analytical study of Taylor instability, Emmons, Chang & Watson (1960) have shown that the growth of the non-sinusoidal surface wave is a non-linear effect. This may very well be true in the present case.

Our aim in this study is therefore to extend the calculation to régimes where the finite amplitude of the surface disturbance can no longer be neglected. We adopt the simple approach of higher order approximation with the initial amplitude as the expansion parameter to study the effects of finite amplitude of the disturbance on the asymmetrical development of the wave-forms.

2. Mathematical formulation

We shall assume in this case that the liquid is inviscid and incompressible. The effect of the surrounding fluid on jet instability is neglected. This will apply, for example, to a moderate speed vertical water jet issuing into atmospheric air. For purpose of analysis we shall assume fluid motion to start from rest so that we have potential flow. The initial axisymmetrical surface disturbance is assumed to be a purely sinusoidal standing wave with amplitude η_0^* and wave-number k^*. For proper interpretation of the results, all physical parameters will be expressed in dimensionless forms. The characteristic length of this problem is R, the radius of the jet in its undisturbed form, and the characteristic time is $(TR^{-3}\rho^{-1})^{-\frac{1}{2}}$, where T is the surface tension and ρ is the liquid density.

The dimensionless velocity potential ϕ and the dimensionless surface disturbance $\eta(z,t)$ in the r-direction must satisfy the following governing equation, boundary and initial conditions.

Governing equation:
$$\nabla^2\phi = 0 \tag{2.1}$$
for
$$r \leqslant 1+\eta \quad \text{and} \quad -\infty < z < \infty.$$

Boundary conditions:
$$-\eta_t+\phi_r-\phi_z\eta_z = 0 \tag{2.2}$$
and
$$1-\phi_t-\tfrac{1}{2}[\phi_z^2+\phi_r^2] = (1+\eta)^{-1}(1+\eta_z^2)^{-\frac{1}{2}}-\eta_{zz}(1+\eta_z^2)^{-\frac{3}{2}} \tag{2.3}$$
on
$$r = 1+\eta(z,t).$$

Initial conditions:
$$\eta(z,0) = \eta_0\cos kz+\eta_0^2(-\tfrac{1}{4})+\eta_0^4(-\tfrac{1}{32})+\dots, \tag{2.4}$$
$$\eta_t(z,0) = 0. \tag{2.5}$$

Equation (2.2) expresses the fact that fluid particles initially on the free surface remain there subsequently. Equation (2.3) shows that the difference in pressure across the free surface is due to surface tension. The initial condition, (2.4), can be obtained as follows.

For an initially sinusoidal disturbance with the undisturbed radius equal to 1, we have
$$r = R+\eta_0\cos kz. \tag{2.4a}$$
For conservation of mass of a column of liquid of length π/k, and volume π^2/k, we have
$$\frac{\pi^2}{k} = \int_0^{\pi/k}\pi(R+\eta_0\cos kz)^2\,dz$$
$$= R^2\frac{\pi^2}{k}+\frac{\eta_0^2}{2}\frac{\pi^2}{k}.$$
Therefore
$$R = \left(1-\frac{\eta_0^2}{2}\right)^{\frac{1}{2}}$$
$$= 1+\eta_0^2(-\tfrac{1}{4})+\eta_0^4(-\tfrac{1}{32})+\dots.$$
Substituting this back into (2.4a), we obtain (2.4).

In order to proceed analytically we shall assume that the surface disturbance can be expressed in the following form:
$$\eta = \sum_{m=1}^{\infty}\eta_0^m\eta_m. \tag{2.6}$$
If η_m and all its derivatives are further assumed to be of the same order of magnitude, it follows from (2.2) that
$$\phi = \sum_{m=1}^{\infty}\eta_0^m\phi_m. \tag{2.7}$$

Since (2.1) is linear, it must be satisfied by each of the ϕ_m separately. The corresponding boundary conditions and initial conditions are obtained by substituting (2.6) and (2.7) into (2.2) to (2.5) and equating equal power of η_0^m. The evaluation of the boundary conditions at $r = 1+\eta(z,t)$ where $\eta(z,t)$ is not known *a priori* is circumvented by expressing ϕ_m at $r = 1+\eta(z,t)$ with a Taylor series expansion at $r = 1$. At each order of the approximation, the boundary conditions are now linear. All non-linear terms involve only lower order functions which at any given order will have been determined.

Proceeding to solve the equations starting with the lowest order, we encountered difficulty in the third-order solutions. The linearized solutions show that the surface wave is stable for $k > 1$. The third-order solutions show that the surface wave is unstable for $k > 1$ which does not seem to agree with available experimental observation. In order to suppress the secular terms in the stable case, we introduce a new variable

$$\tau = \nu t,$$

where
$$\nu = \nu_1 + \eta_0 \nu_2 + \eta_0^2 \nu_3 + \dots.$$

The individual ν is used to eliminate the secular terms from the solution in the stable case of $k > 1$. In the third-order solution only ν_3 is used. The ν_3 determined in this manner is finite except at the cut-off wave-number of $k_c = 1$. That $k_c = 1$ is the boundary to separate the region of stability from instability is only true in the linearized case but not necessarily true in the non-linear case. In order to eliminate the singularity in ν_3, we allow k_c to deviate from 1. The amount of deviation is determined by the condition that the singularity in ν_3 is eliminated at the cut-off wave-number. This method would also apply to any order. To do this we introduce

$$\zeta = k_c z \quad \text{and} \quad \mathscr{K} = k/k_c,$$

where
$$k_c = k_{c1} + \eta_0 k_{c2} + \eta_0^2 k_{c3} + \dots.$$

k_{c1} is equal to 1 in order to agree with the linearized theory.

The equation of motion, initial conditions and boundary conditions (evaluated at $r = 1$) for the first three orders using the new co-ordinates are:

η_0 *order*
$$\nabla^2 \phi_1 = 0, \tag{2.8}$$

Initial condition:
$$\eta_1(\zeta, 0) = \cos \mathscr{K}\zeta, \quad \eta_{1,\tau}(\zeta, 0) = 0, \tag{2.9}$$

Boundary condition:
$$-\nu_1 \eta_{1,\tau} + \phi_{1,r} = 0, \tag{2.10}$$

$$-\nu_1 \phi_{1,\tau} + \eta_1 + \eta_{1,\zeta\zeta} = 0; \tag{2.11}$$

η_0^2 *order*
$$\nabla^2 \phi_2 = -2k_{c2} \phi_{1,\zeta\zeta}, \tag{2.12}$$

Initial condition:
$$\eta_2(\zeta, 0) = -\tfrac{1}{4}, \quad \eta_{2,\tau}(\zeta, 0) = 0, \tag{2.13}$$

Boundary condition:

$$-\nu_1 \eta_{2,\tau} + \phi_{2,r} = -\phi_{1,rr}\eta_1 + \phi_{1,\zeta}\eta_{1,\zeta} + \nu_2 \eta_{1,\tau}, \tag{2.14}$$

$$-\nu_1 \phi_{2,\tau} + \eta_2 + \eta_{2,\zeta\zeta} = \nu_1 \phi_{1,r\tau}\eta_1 + \tfrac{1}{2}[\phi_{1,\zeta}^2 + \phi_{1,r}^2] + \eta_1^2 - \tfrac{1}{2}\eta_{1,\zeta}^2 + \nu_2 \phi_{1,\tau} - 2k_{c2}\eta_{1,\zeta\zeta}; \tag{2.15}$$

η_0^3 *order*
$$\nabla^2 \phi_3 = -[k_{c2}^2 + 2k_{c3}]\phi_{1,\zeta\zeta} - 2k_{c2}\phi_{2,\zeta\zeta}, \tag{2.16}$$

Initial condition:
$$\eta_3(\zeta, 0) = 0, \quad \eta_{3,\tau}(\zeta, 0) = 0, \tag{2.17}$$

Boundary condition:

$$-\nu_1\eta_{3,\tau}+\phi_{3,r} = \nu_2\eta_{2,\tau}+\nu_3\eta_{1,\tau}-\phi_{1,rr}\eta_2-\phi_{2,rr}\eta_1 -\tfrac{1}{2}\phi_{1,rrr}\eta_1^2+[\phi_{2,\zeta}+\phi_{1,r\zeta}\eta_1]\,\eta_{1,\zeta}+\phi_{1,\zeta}\eta_{2,\zeta}+2k_{c2}\phi_{1,\zeta}\eta_{1,\zeta}, \quad (2.18)$$

$$\begin{aligned}-\nu_1\phi_{3,\tau}+\eta_3+\eta_{3,\zeta\zeta} &= \nu_1\phi_{1,r\tau}\eta_2+\nu_1\phi_{2,r\tau}\eta_1+\tfrac{1}{2}\nu_1\phi_{1,rr\tau}\eta_1^2\\ &+\nu_2[\phi_{2,\tau}+\phi_{1,r\tau}\eta_1]+\nu_3\phi_{1,\tau}+\phi_{1,\zeta}[\phi_{2,\zeta}+\phi_{1,r\zeta}\eta_1]+k_{c2}(\phi_{1,\zeta})^2\\ &+\phi_{1,r}[\phi_{2,r}+\phi_{1,rr}\eta_1]+2\eta_1\eta_2-\eta_1^3-\eta_{1,\zeta}\eta_{2,\zeta}+\tfrac{1}{2}\eta_1(\eta_{1,\zeta})^2\\ &-\tfrac{1}{2}kc_2\eta_{1,\zeta}^2+\tfrac{3}{2}\eta_{1,\zeta\zeta}(\eta_{1,\zeta})^2-2k_{c2}\eta_{2,\zeta\zeta}-[k_{c2}^2+2k_{c3}]\,\eta_{1,\zeta\zeta}. \end{aligned} \quad (2.19)$$

3. Solutions

For the first-order approximation, we can set $\nu_1 = 1$. The results are the same as the linearized analysis of Rayleigh:

$$\phi_1 = \frac{\omega_1}{\mathscr{K}}\frac{I_0(\mathscr{K}r)}{I_1(\mathscr{K})}\cos\mathscr{K}\zeta\sinh\omega_1\tau, \quad (3.1)$$

$$\eta_1 = \cos\mathscr{K}\zeta\cosh\omega_1\tau \equiv [B_{11}(\tau)\cos\mathscr{K}\zeta], \quad (3.2)$$

where $$\omega_1^2 = \frac{\mathscr{K}(1-\mathscr{K}^2)}{I_a}, \quad I_a = \frac{I_0(\mathscr{K})}{I_1(\mathscr{K})}.$$

These solutions show that for $\mathscr{K} < 1$, ω_1 is positive and the surface waves grow; for $\mathscr{K} > 1$, ω_1 is purely imaginary, and the surface waves oscillate. The oscillation for $\mathscr{K} > 1$ is due to the assumption of inviscid fluid.

In the second-order approximation, we can set both ν_2 and k_{c2} equal to zero. The method of solution is to assume

$$\eta_2(\zeta,\tau) = B_{22}(\tau)\cos 2\mathscr{K}\zeta+D_2(\tau). \quad (3.3)$$

Substituting into (2.14),

$$\phi_{2,r} = \dot{B}_{22}\cos 2\mathscr{K}\zeta+\dot{D}_2+P_{22}\cos 2\mathscr{K}\zeta\sinh 2\omega_1\tau+\tfrac{1}{4}\omega_1\sinh 2\omega_1\tau, \quad (3.4)$$

where $$P_{22} = \tfrac{1}{4}\omega_1[1-2\mathscr{K}I_a].$$

In order for ϕ_2 to satisfy Laplace's equation and the velocity be finite everywhere, we have to set in (3.4),

$$\dot{D}_2+\tfrac{1}{4}\omega_1\sinh 2\omega_1\tau = 0. \quad (3.5)$$

Now we can construct a solution for ϕ_2 which will satisfy both (2.12) and (3.4), it is

$$\phi_2 = \frac{\dot{B}_{22}}{2\mathscr{K}I_1(2\mathscr{K})}I_0(2\mathscr{K}r)\cos 2\mathscr{K}\zeta+\frac{P_{22}I_0(2\mathscr{K}r)}{2\mathscr{K}I_1(2\mathscr{K})}\cos 2\mathscr{K}\zeta\sinh 2\omega_1\tau+F(\tau). \quad (3.6)$$

Substituting into (2.15), we have two equations to determine B_{22} and $F(\tau)$. The solutions which satisfy the initial conditions [equation (2.13)] are:

$$\eta_2(\zeta,\tau) = B_{22}(\tau)\cos 2\mathscr{K}\zeta-\tfrac{1}{8}(\cosh 2\omega_1\tau+1), \quad (3.7)$$

$$\phi_2(\zeta,\tau) = \frac{\dot{B}_{22}}{2\mathscr{K}I_1(2\mathscr{K})}I_0(2\mathscr{K}r)\cos 2\mathscr{K}\zeta+\frac{P_{22}}{2\mathscr{K}}\frac{I_0(2\mathscr{K}r)}{I_1(2\mathscr{K})}\cos 2\mathscr{K}\zeta\sinh 2\omega_1\tau -\tfrac{1}{8}[3+\omega_1^2(1-I_a^2)]\,\tau-\frac{1}{16\omega_1}[3+\omega_1^2(3+I_a^2)]\sinh 2\omega_1\tau, \quad (3.8)$$

where
$$\omega_2^2 = 2\mathscr{K}(1-4\mathscr{K}^2)/I_b, \quad I_b = I_0(2\mathscr{K})/I_1(2\mathscr{K}),$$
$$B_{22}(\tau) = a_{22}\cosh\omega_2\tau + b_{22}\cosh 2\omega_1\tau + c_{22},$$
$$b_{22} = \frac{2\omega_1^2 I_b(1-2\mathscr{K}I_a)+[2+\omega_1^2(3-I_a^2)]\mathscr{K}}{4I_b(\omega_2^2-4\omega_1^2)},$$
$$c_{22} = \frac{2+\omega_1^2(1+I_a^2)}{8(1-4\mathscr{K}^2)}, \quad a_{22} = -(b_{22}+c_{22}).$$

The equation of the surface, (3.7), shows the growth of the first harmonic and a purely time-dependent term. The purely time-dependent term is required to off-set the addition of volume from the fundamental so that volume is conserved to the second order. The velocity potential shows the first harmonic and a purely time-dependent term which comes partly from the purely time-dependent term in η_2 and also as a direct consequence of the acceleration of the fluid. The growth of the first harmonic is due to two effects, one is the feeding of energy from the fundamental as exemplified by terms having a growth rate of $2\omega_1$, and the other is due to the inherent instability of the first harmonic itself when the dimensionless wave-number $2\mathscr{K} < 1$.

In the third-order approximation, using the results of the first- and second-order solutions, (2.16) becomes

$$\phi_{3,rr}+\frac{1}{r}\phi_{3,r}+\phi_{3,\zeta\zeta} = 2k_{c3}\omega_1\mathscr{K}\frac{I_0(\mathscr{K}r)}{I_1(\mathscr{K})}\cos\mathscr{K}\zeta\sinh\omega_1\tau. \tag{3.9}$$

The particular solution of ϕ_3 is

$$(\phi_3)_p = k_{c3}\omega_1\frac{rI_1(\mathscr{K}r)}{I_1(\mathscr{K})}\cos\mathscr{K}\zeta\sinh\omega_1\tau. \tag{3.10}$$

The boundary conditions [equations (2.18) and (2.19)] can be expressed in the following form:

$$-\eta_{3,\tau}+\phi_{3,r} = P_{31}(\tau)\cos\mathscr{K}\zeta + P_{33}(\tau)\cos 3\mathscr{K}\zeta, \tag{3.11}$$
$$-\phi_{3,\tau}+\eta_3+\eta_{3,\zeta\zeta} = Q_{31}(\tau)\cos\mathscr{K}\zeta + Q_{33}(\tau)\cos 3\mathscr{K}\zeta, \tag{3.12}$$

where P and Q as functions of τ are given in appendix A.

Following the same approach as in the second approximation, we assume

$$\eta_3(\zeta,\tau) = B_{31}(\tau)\cos\mathscr{K}\zeta + B_{33}(\tau)\cos 3\mathscr{K}\zeta. \tag{3.13}$$

Substituting into (3.11),

$$\phi_{3,r} = \dot{B}_{31}(\tau)\cos\mathscr{K}\zeta + \dot{B}_{33}(\tau)\cos 3\mathscr{K}\zeta + P_{31}(\tau)\cos\mathscr{K}\zeta + P_{33}(\tau)\cos 3\mathscr{K}\zeta. \tag{3.14}$$

A solution of ϕ_3 that satisfies both (3.9) and (3.14) is

$$\phi_3 = \left\{[\dot{B}_{31}(\tau)+P_{31}(\tau)-k_{c3}\omega_1\mathscr{K}I_a\sinh\omega_1\tau]\frac{I_0(\mathscr{K}r)}{\mathscr{K}I_1(\mathscr{K})} + k_{c3}\omega_1\frac{rI_1(\mathscr{K}r)}{I_1(\mathscr{K})}\sinh\omega_1\tau\right\}\cos\mathscr{K}\zeta + [\dot{B}_{33}(\tau)+P_{33}(\tau)]\frac{I_0(3\mathscr{K}r)}{3\mathscr{K}I_1(3\mathscr{K})}\cos 3\mathscr{K}\zeta + F_3(\tau). \tag{3.15}$$

Substituting into (3.12) and equating terms having the same harmonic, we have three equations to determine $B_{31}(\tau)$, $B_{33}(\tau)$ and $F_3(\tau)$. In particular, we are interested in $B_{31}(\tau)$. The differential equation for $B_{31}(\tau)$ can be written as

$$\begin{aligned}\ddot{B}_{31}(\tau)-\omega_1^2 B_{31}(\tau) = & -[\mu_1 p_{311}+q_{311}\mathscr{K}/I_a]\cosh\mu_1\tau \\ & -[\mu_2 p_{312}+q_{312}\mathscr{K}/I_a]\cosh\mu_2\tau-[3\omega_1 p_{313}+q_{313}\mathscr{K}/I_a]\cosh 3\omega_1\tau \\ & -[\omega_1 p_{314}+q_{314}\mathscr{K}/I_a-\omega_1^2\mathscr{K}k_{c3}(I_a-1/I_a)]\cosh\omega_1\tau. \qquad (3.16)\end{aligned}$$

The p and q are shown in appendix A.

In order for η_3 to be stable for $\mathscr{K} > 1$, the last term in (3.16) has to be set equal to zero. This determines ν_3 which can be shown to be

$$\begin{aligned}\nu_3 = & \tfrac{1}{8}\gamma(1+\mathscr{K}I_a)-\tfrac{1}{8}\alpha[2+(\mathscr{K}/I_a)]-\tfrac{1}{8}\beta[1-2\mathscr{K}I_b+(\mathscr{K}/I_a)] \\ & +\tfrac{1}{64}[8-5\mathscr{K}I_a+9(\mathscr{K}/I_a)]+\tfrac{1}{2}\mathscr{K}k_{c3}[I_a-(1/I_a)] \\ & +\tfrac{1}{64}(\mathscr{K}/I_a)(1/\omega_1^2)[30-3\mathscr{K}^2-64\mathscr{K}^2k_{c3}+9\mathscr{K}^4]. \qquad (3.17)\end{aligned}$$

The γ, α and β are defined in appendix A. Equation (3.17) shows that ν_3 becomes infinite as ω_1 becomes zero at $\mathscr{K} = 1$. Thus, for ν_3 to be finite at $\omega_1 = 0$, we adjust k_{c3} so that the last term in (3.17) is finite at $\mathscr{K} = 1$. This determines k_{c3} which is

$$k_{c3} = \tfrac{9}{16}. \qquad (3.18)$$

Substituting back into (3.17),

$$\begin{aligned}\nu_3 = & \tfrac{1}{8}\gamma(1+\mathscr{K}I_a)-\tfrac{1}{8}\alpha[2+(\mathscr{K}/I_a)]-\tfrac{1}{8}\beta[1-2\mathscr{K}I_b+(\mathscr{K}/I_a)] \\ & +\tfrac{1}{64}[13\mathscr{K}I_a+38-9\mathscr{K}^2-9\mathscr{K}/I_a]. \qquad (3.19)\end{aligned}$$

We then proceed to solve for $B_{31}(\tau)$, $B_{33}(\tau)$ and $F_3(\tau)$. $F_3(\tau)$ turns out to be a constant and can be neglected. Thus the solution ϕ_3 can be expressed as

$$\begin{aligned}\phi_3(r,\zeta,\tau) = & \left[C_{31}(\tau)\frac{I_0(\mathscr{K}r)}{\mathscr{K}I_1(\mathscr{K})}+k_{c3}\omega_1\frac{rI_1(\mathscr{K}r)}{I_1(\mathscr{K})}\sinh\omega_1\tau\right]\cos\mathscr{K}\zeta \\ & +C_{33}(\tau)\frac{I_0(3\mathscr{K}r)}{3\mathscr{K}I_1(3\mathscr{K})}\cos 3\mathscr{K}\zeta. \qquad (3.20)\end{aligned}$$

The B in (3.13) and C as functions of τ are given in appendix B. The third-order solutions show that as a result of the interaction among the lower harmonics not only a higher harmonic ($\cos 3\mathscr{K}\zeta$) appears but there is also a feedback into the fundamental ($\cos\mathscr{K}\zeta$). The behaviour of the second harmonic is similar to the first harmonic in the sense that above the dimensionless wave-number $3\mathscr{K} > 1$, the growth of the harmonic is sustained only by direct feeding of energy from the lower harmonics.

The solution for the surface wave with an initial harmonic disturbance to third order is

$$\begin{aligned}\eta = & \sum_{m=1}^{\infty}\eta_0^m\eta_m \\ = & \eta_0\cos\mathscr{K}\zeta\cosh\omega_1\tau+\eta_0^2[B_{22}(\tau)\cos 2\mathscr{K}\zeta-\tfrac{1}{8}(\cosh\omega_1\tau+1)] \\ & +\eta_0^3[B_{31}(\tau)\cos\mathscr{K}\zeta+B_{33}(\tau)\cos 3\mathscr{K}\zeta], \qquad (3.21)\end{aligned}$$

where $\tau = \nu t$, $\zeta = k_c z$, $\mathscr{K} = k/k_c$ with $\nu = 1+\eta_0^2\nu_3$, $k_c = 1+\eta_0^2\frac{9}{16}$. This series can be expressed as a Fourier series plus a purely time-dependent series as follows:

$$\eta = \sum_{l=1}^{\infty} A_l(\mathscr{K},\tau)\cos(l\mathscr{K}\zeta) + \sum_{l=1}^{\infty} \eta_0^l D_l(\tau), \tag{3.22}$$

where
$$A_l(\mathscr{K},\tau) = \sum_{m=l}^{\infty} B_{ml}(\mathscr{K},\tau)\,\eta_0^m$$

and the B_{ml} are given in appendix B up to B_{33}. The $D_l(\tau)$ for $l = 2$ is given as the last term in (3.7). Thus, the Fourier series displays the distortion of the wave-form whereas the purely time-dependent series is required for the conservation of mass.

4. Results and discussion

Typical results of (3.21) for $k = 0{\cdot}95$, $0{\cdot}7$ and $0{\cdot}3$ and $\eta_0 = 0{\cdot}01$ are shown in figures 1–3. They show the asymmetrical developments of the wave profiles in time at various dimensionless wave-numbers. For short wavelengths, the neck

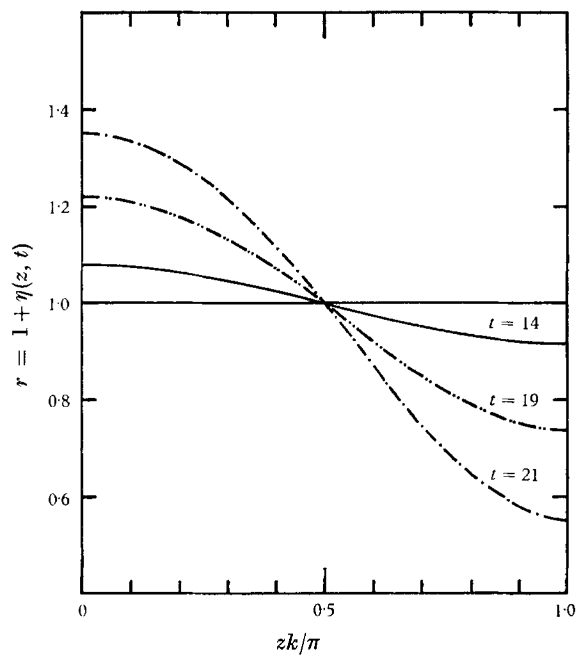

FIGURE 1. Calculated wave profiles at different times for $k = 0{\cdot}95$ and $\eta_0 = 0{\cdot}01$. The vertical scale is magnified by a factor of π/k with respect to the horizontal scale.

grows faster than the swell as shown in figure 1. The nodal point at $kz = \frac{1}{2}\pi$ which, according to the linearized theory, will stay at $r = 1$ remains relatively stationary. For $k = 0{\cdot}7$ as shown in figure 2, a similar trend exists but the nodal point is moving inward. As discussed previously, for the sinusoidal wave-form in (1·2) to hold, the motion of R which is equivalent to the motion of the nodal point and represented by the $D_2(t)$ term must be a decreasing function of time. For $k = 0{\cdot}7$ with $t = 13$,

$$\eta(0,13)-\eta(\tfrac{1}{2}\pi,13) = 0{\cdot}457 \quad \text{and} \quad \eta(\pi,13)-\eta(\tfrac{1}{2}\pi,13) = -0{\cdot}438$$

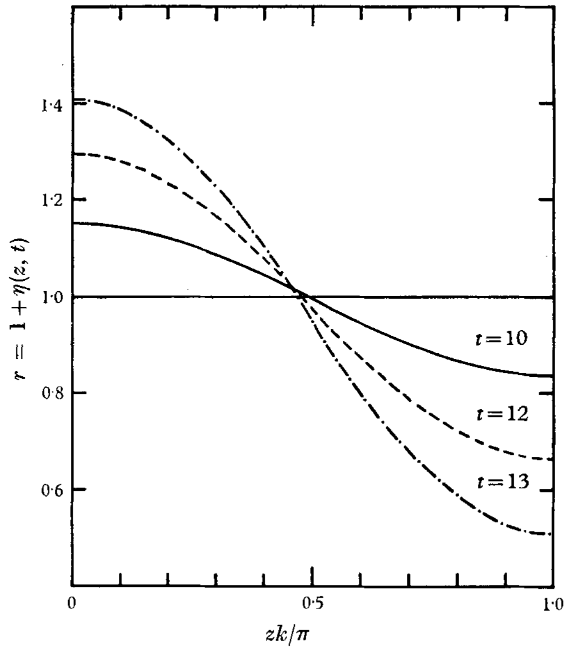

FIGURE 2. Calculated wave profiles at different times for $k = 0{\cdot}7$ and $\eta_0 = 0{\cdot}01$. The vertical scale is magnified by a factor of π/k with respect to the horizontal scale.

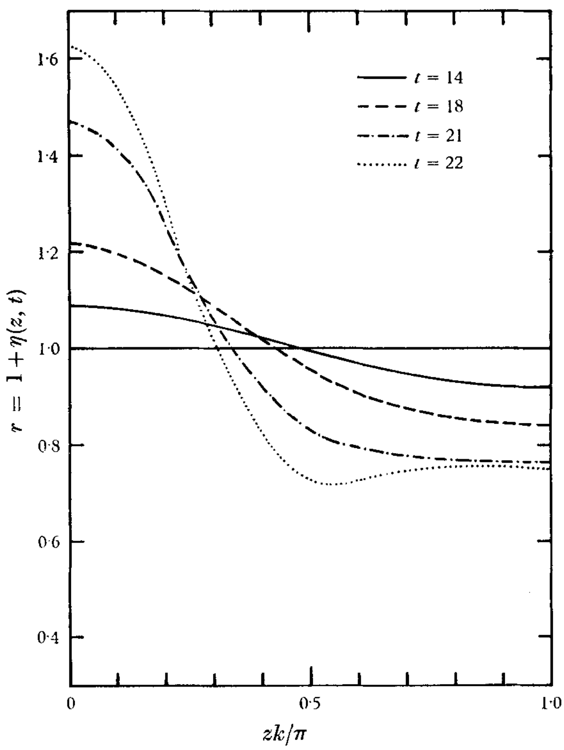

FIGURE 3. Calculated wave profiles at different times for $k = 0{\cdot}3$ and $\eta_0 = 0{\cdot}01$. The vertical scale is magnified by a factor of π/k with respect to the horizontal scale.

showing that the sinusoidal shape is preserved to a large extent. Figure 3 shows that for longer wavelength at $k = 0{\cdot}3$, the trend reverses so that the swell grows faster than the neck and the motion of the nodal point is substantial. The result is that the asymmetry is striking with a narrowing of the swell and a broadening of the neck of the wave.

k	a_{22}	b_{22}	a_{31}	b_{31}	c_{31}	d_{31}
0·95	0·260	**−0·109**	−0·498	−0·0067	0·0638i	**0·507**
0·7	0·468	**0·0121**	0·00673	−0·0753	0·340i	**0·0685**
0·3	**−1·81**	**1·04**	**−1·79**	**2·18**	**−0·406**	**−0·387**

k	a_{33}	b_{33}	c_{33}	d_{33}	e_{33}	ν_3
0·95	0·0673	−0·114	0·0597i	**0·00915**	0·0375	0·875
0·7	0·125	−0·223	0·259i	**−0·0279**	0·127	1·45
0·3	**10·56**	**−13·8**	**11·2**	**0·537**	2·66	0·302

TABLE 1. Numerical values of coefficients of amplitude functions and ν_3 at $k = 0{\cdot}95$, $0{\cdot}7$ and $0{\cdot}3$. (c_{22} which is unimportant is omitted.)

To pursue these points further, we present in table 1 the coefficients in B_{22}, B_{31} and B_{33} for $k = 0{\cdot}3$, $0{\cdot}7$ and $0{\cdot}95$. The most important coefficients for t not small are bold face. For example, at $k = 0{\cdot}7$, d_{31} is much more important than b_{31} because the time-dependent term associated with d_{31} is cosh $3\omega_1 t$, whereas that of b_{31} is only $\cos|\omega_2|t \cosh \omega_1 t$. It is seen that b_{22} in second order and b_{33} in third order are responsible for the distortion of the original sinusoidal wave form up to third order. Table 1 shows that the coefficient of b_{22} is smallest at $k = 0{\cdot}7$ and b_{33} is small for both $k = 0{\cdot}7$ and $0{\cdot}95$, thus demonstrating that the distortion of the wave-form is minimum at around $k = 0{\cdot}7$. Although the b_{31} term will not distort the wave, it nevertheless changes the growth rate from that predicted by the linearized theory. Its effect seems to be smallest at $k = 0{\cdot}7$. For $k = 0{\cdot}95$, the effect of the $D_2(t)$ term which is to contract the jet radius uniformly is opposed by the b_{22} term at the nodal point, thus making the nodal point almost stationary as shown in figure 1. These calculated surface profiles agree qualitatively with all the available experimental observations including the photographs of Donnelly & Glaberson. These figures plus further analysis of (3.21) show that the surface wave grows fastest at $k = 0{\cdot}7$, which is in agreement with the linearized theory.

The present third-order theory seems to describe well the early stages of the growth of an initially sinusoidal wave. The exact limitation of the present theory is difficult to specify without further knowledge of the behaviour of the still higher order terms. However, figure 3 indicates that for $k = 0{\cdot}3$, $\eta_0 = 0{\cdot}01$ and $t = 22$, the surface profile shows undulation near $kz = \frac{1}{2}\pi$ which is contrary to experimental observation, indicating a breakdown of the present theory. For higher k where table 1 shows the non-linear effect is less important, the present theory will probably be adequate up to $\eta = 0{\cdot}5$ if $\eta_0 = 0{\cdot}01$.

The effects of the initial amplitudes of the disturbance on the development of wave-forms are shown in figure 4. As expected, asymmetry of the surface disturbance occurs much earlier for initial disturbances of large amplitude than for those of small amplitude.

The present third-order theory shows that for an initial sinusoidal wave-form with a dimensionless wave-number k, the fundamental frequency of oscillation (or growth rate for the unstable case) is different from the linearized theory by a factor of $1+\eta_0^2\nu_3$. The numerical values of ν_3 are of order unity as shown in table 1. The dimensionless cut-off wave-number k_c, which separates the region of stability

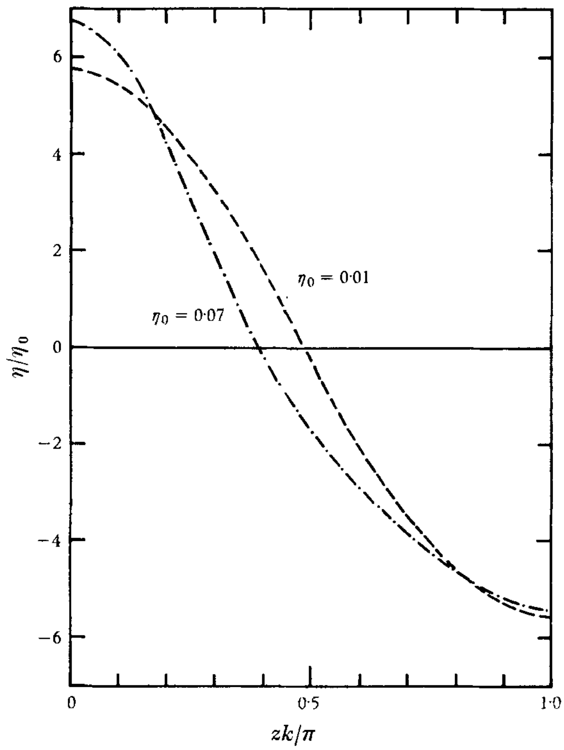

FIGURE 4. Calculated wave profiles with different initial amplitudes for $k = 0{\cdot}3$ and $t = 12{\cdot}0$.

from instability, is equal to $1+\frac{9}{16}\eta_0^2$ in contrast to 1 for the linearized case. The linearized theory further predicts that the surface wave becomes stationary at the cut-off wave-number, but analysis of the third-order theory shows that it still experiences a growth at $k = k_c$. The growth term is

$$\eta = \eta_0^3 \frac{3t^2}{16I_a}.$$

The above prediction has not been observed experimentally. The difficulty seems to be that the growth rate at $k = k_c$ is linearly proportional to t (exponential for $k < k_c$). Thus, for a moderate speed liquid jet of the order of a metre long, t in general is not large enough for the third-order growth to be observed.

Finally, we would like to address ourselves to the question of the excellent agreement between Donnelly & Glaberson's experimental results and Rayleigh's linearized theory of jet instability. Following their method of measurement which is equivalent to subtracting η at $kz = \pi$ (neck) from η at $kz = 0$ (swell) in (3.21), the second-order terms cancel out. Thus in their method of measurement, the correction from the linearized theory is only in third order. One may generalize that for any growth of an initial sinusoidal disturbance where the distortion is due to higher harmonics, the above method of measurement is an excellent way to measure the linearized growth rate.

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Appendix A

Amplitudes of forcing functions of the third-order equations (3.11) and (3.12):

$$P_{31}(\tau) = p_{311}\sinh\mu_1\tau + p_{312}\sinh\mu_2\tau + p_{313}\sinh 3\omega_1\tau + p_{314}\sinh\omega_1\tau,$$
$$P_{33}(\tau) = p_{331}\sinh\mu_1\tau + p_{332}\sinh\mu_2\tau + p_{333}\sinh 3\omega_1\tau + p_{334}\sinh\omega_1\tau,$$
$$Q_{31}(\tau) = q_{311}\cosh\mu_1\tau + q_{312}\cosh\mu_2\tau + q_{313}\cosh 3\omega_1\tau + q_{314}\cosh\omega_1\tau,$$
$$Q_{33}(\tau) = q_{331}\cosh\mu_1\tau + q_{332}\cosh\mu_2\tau + q_{333}\cosh 3\omega_1\tau + q_{334}\cosh\omega_1\tau,$$

with $\mu_1 = \omega_2+\omega_1,\quad \mu_2 = \omega_2-\omega_1,\quad \mu_3 = \omega_2^2+\omega_1^2,\quad \mu_4 = \omega_2^2-\omega_1^2,$

$$\beta = 2b_{22}+\frac{P_{22}}{\omega_1},\quad \alpha = b_{22}+2c_{22},\quad \gamma = b_{22}-2c_{22},$$
$$I_c = I_1(3\mathscr{K})/I_0(3\mathscr{K}),\quad J = 1-3\mathscr{K}I_a,\quad L = 1+I_aI_b,$$
$$\omega_3^2 = 3\mathscr{K}(1-9\mathscr{K}^2)I_c,$$

$$p_{311} = [\mathscr{K}(\omega_1 I_a-\omega_2 I_b)+\mu_1]\,a_{22}/4,$$
$$p_{312} = -[\mathscr{K}(\omega_1 I_a+\omega_2 I_b)-\mu_2]\,a_{22}/4,$$
$$p_{313} = [8\omega_1 b_{22}(3-2\mathscr{K}I_b+\mathscr{K}I_a)+8P_{22}(1-\mathscr{K}I_b)-\omega_1(8+\mathscr{K}^2)+5\omega_1\mathscr{K}I_a]/32,$$
$$p_{314} = [8\beta(1-\mathscr{K}I_b)-8\gamma(1+\mathscr{K}I_a)+5\mathscr{K}I_a-8-\mathscr{K}^2+32\nu_3]\,\omega_1/32,$$
$$p_{331} = [\mu_1-3\mathscr{K}(\omega_1 I_a+\omega_2 I_b)]\,a_{22}/4,$$
$$p_{332} = [\mu_2+3\mathscr{K}(\omega_1 I_a-\omega_2 I_b)]\,a_{22}/4,$$
$$p_{333} = [8b_{22}J+8\beta(1-3\mathscr{K}I_b)+\mathscr{K}I_a-3\mathscr{K}^2-2]\,\omega_1/32,$$
$$p_{334} = [8\beta(1-3\mathscr{K}I_b)-8\gamma J-2-3\mathscr{K}^2+\mathscr{K}I_a]\,\omega_1/32.$$
$$q_{311} = [\mu_3+\omega_1\omega_2 L+2-2\mathscr{K}^2]\,a_{22}/4,$$
$$q_{312} = [\mu_3-\omega_1\omega_2 L+2-2\mathscr{K}^2]\,a_{22}/4,$$
$$q_{313} = [8b_{22}(2-2\mathscr{K}^2+\omega_1^2)+8\omega_1^2\beta(2+L)+11\omega_1^2(\mathscr{K}I_a-1)-10-3\mathscr{K}^4+\mathscr{K}^2]/32,$$
$$q_{314} = [8\alpha(\omega_1^2+2-2\mathscr{K}^2)+8\omega_1^2\beta(2-L)$$
$$+\omega_1^2(\mathscr{K}I_a-9+32[\nu_3/\mathscr{K}]\,I_a)-30-9\mathscr{K}^4+3\mathscr{K}^2+64k_{c3}\mathscr{K}^2]/32,$$
$$q_{331} = [\mu_1^2-\omega_1\omega_2 L+2+2\mathscr{K}^2]\,a_{22}/4,$$
$$q_{332} = [\mu_2^2+\omega_1\omega_2 L+2+2\mathscr{K}^2]\,a_{22}/4,$$
$$q_{333} = [8b_{22}(2+2\mathscr{K}^2+\omega_1^2)+8\omega_1^2\beta(4-L)+\omega_1^2(\mathscr{K}I_a-3)-2+3\mathscr{K}^4-\mathscr{K}^2]/32,$$
$$q_{334} = [8\alpha(\omega_1^2+2+2\mathscr{K}^2)+8\omega_1^2\beta L-\omega_1^2 J-3(2-3\mathscr{K}^4+\mathscr{K}^2)]/32.$$

Appendix B

Amplitude functions of equations (3.13) and (3.20):

$$B_{31}(\tau) = a_{31}\cosh\omega_1\tau + b_{31}\cosh\omega_1\tau\cosh\omega_2\tau + c_{31}\sinh\omega_1\tau\sinh\omega_2\tau + d_{31}\cosh 3\omega_1\tau,$$

$$B_{33}(\tau) = a_{33}\cosh\omega_3\tau + b_{33}\cosh\omega_1\tau\cosh\omega_2\tau + c_{33}\sinh\omega_1\tau\sinh\omega_2\tau + d_{33}\cosh 3\omega_1\tau + e_{33}\cosh\omega_1\tau,$$

$$C_{31}(\tau) = \dot{B}_{31}(\tau) + P_{31}(\tau) - k_{c3}\omega_1\mathscr{K}I_a\sinh\omega_1\tau,$$

$$C_{33}(\tau) = \dot{B}_{33}(\tau) + P_{33}(\tau),$$

$$a_{31} = -(b_{31}+d_{31}),$$

$$b_{31} = [\mathscr{K}(I_a\omega_1^2 + I_b\omega_2^2) + 3\omega_1^2 - \omega_2^2 - (\mu_4 + 2 - 2\mathscr{K}^2)\mathscr{K}I_a^{-1}]\,a_{22}[2(\omega_2^2 - 4\omega_1^2)]^{-1},$$

$$c_{31} = [\mathscr{K}I_a(\omega_1^2 - \mu_4) - 2\mathscr{K}I_b\omega_2^2 + 2\omega_1^2 + (4 - 4\mathscr{K}^2 + \omega_1^2 + \mu_3)\mathscr{K}I_a^{-1}] \times a_{22}\omega_1[2\omega_2(\omega_2^2 - 4\omega_1^2)]^{-1},$$

$$d_{31} = [2\mathscr{K}I_b - 3 - 3\mathscr{K}I_a^{-1}]\beta(32)^{-1} - [24(b_{22}-1) + 8\mathscr{K}^2 + 3\mathscr{K}I_a(8b_{22}+5) + (8b_{22}-11)\mathscr{K}I_a^{-1}](256)^{-1} + [10 + 3\mathscr{K}^4 - \mathscr{K}^2 - 16b_{22}(1+\mathscr{K}^2)] \times\mathscr{K}(256I_a\omega_1^2)^{-1},$$

$$a_{33} = -[b_{33} + d_{33} + e_{33}],$$

$$b_{33} = \{-3\mathscr{K}I_a\omega_1^2(\mu_4+\omega_3^2) + 3\mathscr{K}I_b\omega_2^2(\mu_4-\omega_3^2) + (\omega_3^2\mu_3 - \mu_4^2)(1+3\mathscr{K}I_c) + 6\mathscr{K}I_c[(1+\mathscr{K}^2)(\omega_3^2-\mu_3) - \omega_2^2\omega_1^2L]\}\left\{\frac{a_{22}}{2[(\omega_3^2-\mu_3)^2 - 4\omega_1^2\omega_2^2]}\right\},$$

$$c_{33} = \{3\mathscr{K}I_a(\mu_4-\omega_3^2) - 3\mathscr{K}I_b(\mu_4+\omega_3^2) + 2\omega_3^2 + 3\mathscr{K}I_c[2\omega_3^2 + 4(1+\mathscr{K}^2) + L(\mu_3-\omega_3^2)]\}\left\{\frac{a_{22}\omega_2\omega_1}{2[(\omega_3^2-\mu_3)^2 - 4\omega_1^2\omega_2^2]}\right\},$$

$$d_{33} = \{8\omega_1^2\beta[1 - \mathscr{K}(3I_b - 3I_c + I_aI_bI_c)] + \omega_1^2[\mathscr{K}I_a - 3\mathscr{K}^2 - 2 + 8b_{22}(J+\mathscr{K}I_c) + \mathscr{K}I_c(\mathscr{K}I_a - 3)] - \mathscr{K}I_c[2 - 3\mathscr{K}^4 + \mathscr{K}^2 - 16b_{22}(1+\mathscr{K}^2)]\}\left\{\frac{3}{32(\omega_3^2 - 9\omega_1^2)}\right\},$$

$$e_{33} = \frac{1}{32(\omega_3^2-\omega_1^2)}\{8\omega_1^2\beta[1 + 3\mathscr{K}(I_cL - I_b)] + \omega_1^2[\mathscr{K}I_a - J(8\gamma + 3\mathscr{K}I_c) - 2 - 3\mathscr{K}^2] + 3\mathscr{K}I_c[8\alpha(2 + 2\mathscr{K}^2 + \omega_1^2) - 3(2 - 3\mathscr{K}^4 + \mathscr{K}^2)]\}.$$

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